J Math Chem (2010) 47:856–870
DOI 10.1007/s10910-009-9605-6

ORIGINAL PAPER

Proof of the Fukui conjecture via resolution of singularities and related methods: III

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Received: 27 August 2009 / Accepted: 31 August 2009 / Published online: 19 September 2009

Abstract The present article is a direct continuation of the second part of this series. In conjunction with the analysis of the energy band curves of carbon nanotubes, we develop here fundamental theoretical tools, which are essential to prove the Local Analyticity Proposition (LAP). The LAP enables one to prove the Fukui conjecture (the guiding conjecture for developing the repeat space theory) in a new and powerful context of the theory of algebraic curves and resolution of singularities. The present fundamental tools also serve as modular tools for the repeat space theory, by which one can solve a variety of additivity and molecular network problems in a unifying manner.

Keywords Fukui conjecture · Repeat space theory (RST) · Additivity problems · Asymptotic linearity theorem (ALT) · Resolution of singularities

1 Introduction

The repeat space theory [1–11] and its guiding conjecture, the Fukui conjecture (cf. [1–3] and references therein) played a pivotal role in the First Generation Fukui Project. This international and interdisciplinary joint research project, initiated by the late Professor Kenichi Fukui in 1992, has been going on as the Second Generation Fukui Project in conjunction with the theories developed in the present article and articles [2–6].

This article is dedicated to the memory of the late Professors Kenichi Fukui and Haruo Shingu.

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In the previous part II of this series of articles, we have established the following sequence of logical implications:

LAP1 ⇒ Functional Asymptotic Linearity Theorem ⇒ the Fukui conjecture,

where LAP1 is given by the following and shall be referred to as the Target Proposition in the present part III and the forthcoming part IV of this series of articles:

Proposition 1.1 (Target Proposition (Local Analyticity Proposition, version 1, LAP1)). *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and let* $I = [a, b]$. *Let* $p \in C^{\omega}(I)[\lambda]$ *be a monic polynomial of degree* $q \in \mathbb{Z}^+$ *given by*

$$p = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q.$$

Suppose that for any $\theta \in I$, *the polynomial*

$$\mathrm{Ev}_\theta\,(p) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta)$$

over the field $\mathbb{R}$ *has* q *real roots.*

Define $f : I \times \mathbb{R} \to \mathbb{R}$ by

$$f(\theta, \lambda) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta).$$

Then, for any $(\theta, \lambda) \in f^{-1}(0) \cap (]a, b[\times \mathbb{R})$ there exist $\varepsilon, \delta > 0, n \in \mathbb{Z}^+$, and $h_1, \ldots, h_n \in H_r(\theta - \varepsilon, \theta + \varepsilon)$ with $h_1(\theta) = \cdots = h_n(\theta) = \lambda$ such that

$$f^{-1}(0) \cap (]\theta - \varepsilon, \theta + \varepsilon[\times]\lambda - \delta, \lambda + \delta[) = \bigcup_{i=1}^{n} \Gamma(h_i).$$

The purpose of the present part III of this series is to construct fundamental theoretical tools which will be used for the proof of the Target Proposition in part IV of this series. We shall here construct the fundamental tools in such a way that they are directly associated with the ‘energy band curves’ used in articles [2,3]. We remark that the energy band curves given in article [2] are expressed in terms of the zeroes $f^{-1}(0)$ of the function f given in the above Target Proposition. In reading the following sections (and the forthcoming part IV of this series), the reader is referred to ref. [2] for energy band curves of carbon nanotubes in order to see how the abstract notions from the field of algebraic curves and resolution of singularities are used to tackle molecular problems that originated in chemistry.

2 Fundamental tools for the target proposition and the RST

In this part III and in the forthcoming part IV of this series, for the proof of the Target Proposition, we construct fundamental theoretical tools, in such a way that they are also readily usable as modular parts of the repeat space theory (RST) [1–11], which has a variety of applications in chemistry. Bearing both these imminent and long-range purposes in mind, here in this part III, we concentrate on developing tools related to the parameterization and Puiseux series expansion of analytic curves, which are essential for our purposes.

In Sect. 3, we develop 2 fundamental tools, Propositions 3.1 and 3.2, which are related to the parameterization for each irreducible branch of an analytic curve [12].

In Sect. 4, we develop 5 fundamental tools, Propositions 4.1, 4.2, 4.3.A, 4.3.B, and 4.4, which are related to the Puiseux series expansion of each irreducible branch of an analytic curve (cf. [12] and references therein).

Fukui's DNA problem is a long-range target of the (First and Second Generation) Fukui Project, whose underlying motive has been to cultivate a new interdisciplinary region between chemistry and mathematics for a future development of theoretical chemistry. "Can the conductivity and other properties of a single-walled carbon nanotube be analyzed in the setting of a *-algebra equipped with a complete metric?" This metric problem is fundamental to proceed towards a solution of Fukui's DNA problem. This metric problem has been affirmatively solved in recent article [4], entitled 'Normed repeat space and its super spaces: fundamental notions for the second generation Fukui project'. The reader is referred to article [4] for the notion of the normed repeat space, which will be used (together with fundamental theoretical tools developed in the present series) in the Second Generation Fukui Project.

3 Fundamental tools I

Let $\mathbb{Z}^+$, $\mathbb{Z}_0^+$, $\mathbb{Z}$, $\mathbb{R}^+$, $\mathbb{R}_0^+$, $\mathbb{R}$ and $\mathbb{C}$ denote respectively, the set of all positive integers, nonnegative integers, integers, positive real numbers, nonnegative real numbers, real numbers, complex numbers. Throughout, we retain the notation employed in the preceding parts I and II of this series of articles [5,6]. (The reader is referred to refs. [5,6] for the notation.) The new notation we need in this article is given as follows:

Notation 3.1 For $x \in \mathbb{C}$ and $r \in \mathbb{R}^+$, let

$$\Delta_x(r) := \{y \in \mathbb{C} : |y - x| < r\}. \tag{3.1}$$

For $x \in \mathbb{C}$, let

$$\Delta_x(\infty) := \mathbb{C}. \tag{3.2}$$

For $r \in \mathbb{R}^+$, let

$$\Delta(r) := \Delta_0(r). \tag{3.3}$$

$\mathbb{R}\{z\}$: the ring of all convergent power series with real coefficients in the indeterminate z.

For each $\psi \in \mathbb{R}\{z\}$, $\rho(\psi)$ denotes the radius of convergence of ψ.

$\mathbb{C}\{z\}$: the ring of all convergent power series with complex coefficients in the indeterminate z.

For each $\psi \in \mathbb{C}\{z\}$, $\rho(\psi)$ denotes the radius of convergence of ψ.

$H(G)$: the ring of all complex-valued analytic functions defined on an open subset G of $\mathbb{C}$.

$H_0(\Delta(r)) \ := \ \{f \in H(\Delta(r)) : \exists\psi \in \mathbb{C}\{z\}$with the radius of convergence $\rho(\psi) \geq r$ such that $f(x) = \psi(x)$ for all $x \in \Delta(r)\}$.

If $a, b \in \mathbb{R}\, with\, a < b$, let $H_r\,(a, b)$ denote the set of all real-valued real analytic functions defined on the interval $]a, b[$.

For $x \in \mathbb{C} - \{0\}$, let Arg x denote the unique real number

$\theta \in [0, 2\pi[$ such that

$$x = |x|\exp(i\theta). \tag{3.4}$$

For $k \in \mathbb{Z}^+$ and $j \in \{1, \ldots, k\}$, define $\hat{\nu}_{k,j} : \mathbb{C} \to \mathbb{C}$ by

$$\hat{\nu}_{k,j}(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ |x|^{1/k} \exp\left(i\left(\text{Arg}\, x + 2\pi j\right)/k\right) & \text{if} \quad x \neq 0. \end{cases} \tag{3.5}$$

If X_0 and X are nonempty sets with $X_0 \subset X$, $i : X_0 \to X$ denotes the inclusion mapping, i.e., i is the mapping of X_0 into X defined by $i(x) = x$ for all $x \in X_0$.

Let $r_1 \in \mathbb{R}^+$ and let $f \in H(\Delta(r_1))$. Suppose that $f(0) = 0$ and that f is not identically zero in a neighborhood of the origin. Let $k \in \mathbb{Z}^+$ be the order of the multiplicity of the zero of f at the origin. Then the Taylor expansion of $w = f(t)$ at the origin is of the form:

$$w = ct^k\,(1 + f_1(t)), \tag{3.6}$$

where $c \in \mathbb{C} - \{0\}$ and the function f_1 is analytic at the origin and vanishes at the origin: $f_1(0) = 0$. Selecting one of the branches of $(\cdot)^{1/k}$, and fixing a suitable $r \in\,]0, r_1]$, define the function $\hat{h}_1 \in H\,(\Delta(r))$ by

$$\hat{h}_1(t) = \hat{\nu}_{k,k}\,(c)\,t\,(1 + f_1(t))^{1/k} \tag{3.7}$$

so that $\hat{h}_1$ is an injection and its inverse function defined on $\hat{h}_1\,(\Delta(r))$ is analytic. (By using the analytic version of the Implicit Function Theorem, one easily verifies that there exists such an $r \in\,]0, r_1]$.) Define functions h, h_1, h_2, and μ by

$$h(t) = f(t), \tag{3.8}$$

$$h_1(t) = \hat{h}_1(t), \tag{3.9}$$

$$h_2(t) = t^k, \tag{3.10}$$

$$\mu(t) = h_1^{-1}(t). \tag{3.11}$$

Then the following diagram 1 is clearly commutative.

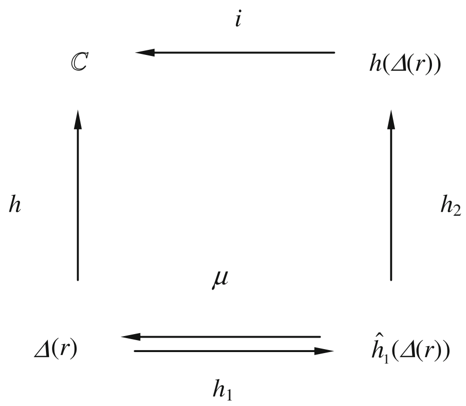

Diagram 1

Recall the following theorem and note that both $h\,(\Delta(r))$ and $\hat{h}_1\,(\Delta(r))$ are open subsets of $\mathbb{C}$ containing the origin, hence that they are both neighborhoods of the origin.

Open Mapping Theorem Let $G \subset \mathbb{C}$ be a region and suppose that f is a nonconstant analytic function on G. Then for any open set U in G, $f(U)$ is open.

Remark Cf. e.g. [13,14].

Since $h\,(\Delta(r))$ and $\hat{h}_1\,(\Delta(r))$ are neighborhoods of the origin, there exists an $s > 0$ such that

$$\Delta(s) \subset h\,(\Delta(r)), \tag{3.12}$$

$$\Delta(s^{1/k}) \subset \hat{h}_1\,(\Delta(r))\,. \tag{3.13}$$

Fix such an $s > 0$. For each $j \in \{1, \ldots, k\}$, let $\nu_j : \Delta(s) \to \Delta\left(s^{1/k}\right)$ be the function defined by

$$\nu_j(x) = \hat{\nu}_{k,j}(x) \tag{3.14}$$

Then for each $j \in \{1, \ldots, k\}$, the following diagram 2 is commutative.

Remark If k is not equal to 1, for each j, the mapping ν_j is not continuous entirely on its domain, but this does not affect our argument in the present paper. Notice that the mapping h_1 and $\hat{h}_1$ are different from each other, since their targets are different although their graphs are exactly the same. The h_1 is a bijection, whereas the complex-valued function $\hat{h}_1$ is not generally surjective.

Now we are ready to state and prove the first two fundamental propositions, which are related to the parameterization for each irreducible branch of an analytic curve:

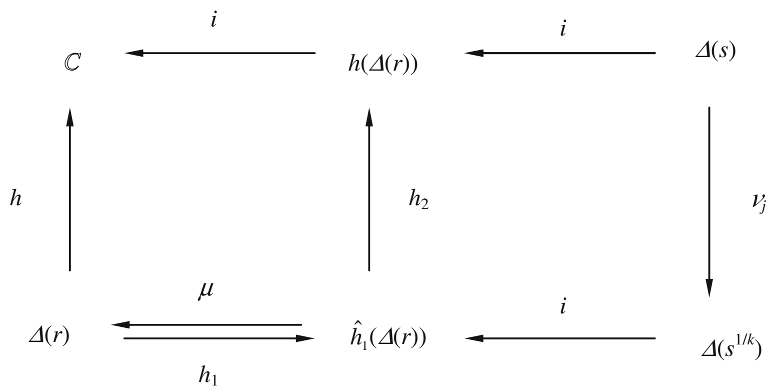

Diagram 2

Proposition 3.1 *The notation and the assumption being as above, we have*

(i) $\left\{z \in \mathbb{C} : z^k - x = 0\right\} = \left\{\hat{\nu}_{k,1}(x), \ldots, \hat{\nu}_{k,k}(x)\right\}$.
(ii) $x \in \Delta(s)$ *implies that* $h_2^{-1}(\{x\}) = \{\nu_1(x), \ldots, \nu_k(x)\}$.
(iii) $x \in \Delta(s)$ *implies that* $h^{-1}(\{x\}) = \{\mu(\nu_1(x)), \ldots, \mu(\nu_k(x))\}$.

Proof (i) The assertion directly follows from the fact that

$$z^k - x = 0 \Leftrightarrow \left(z - \hat{\nu}_{k,1}(x)\right) \ldots \left(z - \hat{\nu}_{k,k}(x)\right) = 0. \tag{3.15}$$

(ii) Let $x \in \Delta(s)$. The commutativity of the above diagram 2 for all $j \in \{1, \ldots, k\}$ implies that

$$h_2\left(\nu_j(x)\right) = x \tag{3.16}$$

for all $j \in \{1, \ldots, k\}$. Hence

$$\nu_j(x) \in h_2^{-1}(\{x\}) \tag{3.17}$$

for all $j \in \{1, \ldots, k\}$, i.e.,

$$h_2^{-1}(\{x\}) \supset \{\nu_1(x), \ldots, \nu_k(x)\}. \tag{3.18}$$

By using (i), the reverse inclusive relation can be obtained as follows:

$$\begin{aligned} h_2^{-1}(\{x\}) &= \left\{z \in \hat{h}_1(\Delta(r)) : z^k - x = 0\right\} \\ &\subset \left\{z \in \mathbb{C} : z^k - x = 0\right\} \\ &= \left\{\hat{\nu}_{k,1}(x), \ldots, \hat{\nu}_{k,k}(x)\right\} \\ &= \{\nu_1(x), \ldots, \nu_k(x)\}. \end{aligned} \tag{3.19}$$

(iii) The conclusion readily follows from the commutativity of diagram 1 and from (ii):

$$\begin{aligned} h^{-1}(\{x\}) &= (h_2 \circ h_1)^{-1}(\{x\}) \\ &= \mu\left(h_2^{-1}(\{x\})\right) \\ &= \mu(\{\nu_1(x), \ldots, \nu_k(x)\}) \\ &= \{\mu(\nu_1(x)), \ldots, \mu(\nu_k(x))\}. \end{aligned} \tag{3.20}$$

□

Proposition 3.2 *The notation and the assumption being as above, let g be any complex-valued function defined on* $\Delta(r_1)$. *Then the following equality holds:*

$$\{(f(t), g(t)) : t \in \Delta(r)\} \cap (\Delta(s) \times \mathbb{C}) = \bigcup_{j=1}^{k} \left\{\left(x, g\left(\mu\left(\nu_j(x)\right)\right)\right) : x \in \Delta(s)\right\}. \tag{3.21}$$

Proof Recalling the definition of h, $h := f|\Delta(r)$, we easily see that

$$\begin{aligned} &\{(f(t), g(t)) : t \in \Delta(r)\} \cap (\Delta(s) \times \mathbb{C}) \\ &= \{(x, y) \in \Delta(s) \times \mathbb{C} : \exists t \in \Delta(r)\,[x = f(t) \wedge y = g(t)]\} \\ &= \{(x, y) \in \Delta(s) \times \mathbb{C} : \exists t \in \Delta(r)\,[x = h(t) \wedge y = g(t)]\} \\ &= \left\{(x, y) \in \Delta(s) \times \mathbb{C} : \exists t \in \Delta(r)\left[t \in h^{-1}(\{x\}) \wedge y = g(t)\right]\right\} \end{aligned} \tag{3.22}$$

Note that if $(x, y) \in \Delta(s) \times \mathbb{C}$ then the following statements are equivalent:

(S1) $\exists t \in \Delta(r)\left[t \in h^{-1}(\{x\}) \wedge y = g(t)\right]$.

(S2) $\exists t \in \Delta(r)\left[\left(\bigvee_{j=1}^{k}\left[t = \mu\left(\nu_j(x)\right)\right]\right) \wedge y = g(t)\right]$.

(S3) $\exists t \in \Delta(r)\left[\bigvee_{j=1}^{k}\left[t = \mu\left(\nu_j(x)\right) \wedge y = g(t)\right]\right]$.

(S4) $\bigvee_{j=1}^{k}\left[\exists t \in \Delta(r)\left[t = \mu\left(\nu_j(x)\right) \wedge y = g(t)\right]\right]$.

(S5) $\bigvee_{j=1}^{k}\left[y = g\left(\mu\left(\nu_j(x)\right)\right)\right]$.

Here, (S1) ⇔ (S2) follows from Proposition 3.1 (iii); (S2) ⇔ (S3) ⇔ (S4) follows from the fundamental properties of $\wedge$ and $\vee$; and (S4) ⇔ (S5) follows from the fact that the target of the function μ is $\Delta(r)$.

Thus, we have

$$\begin{aligned} &\{(f(t), g(t)) : t \in \Delta(r)\} \cap (\Delta(s) \times \mathbb{C}) \\ &= \left\{(x, y) \in \Delta(s) \times \mathbb{C} : \bigvee_{j=1}^{k}\left[y = g\left(\mu\left(\nu_j(x)\right)\right)\right]\right\} \end{aligned}$$

$$= \bigcup_{j=1}^{k} \left\{ (x, y) \in \Delta(s) \times \mathbb{C} : y = g\left(\mu\left(\nu_j(x)\right)\right) \right\}$$
$$= \bigcup_{j=1}^{k} \left\{ \left(x, g\left(\mu\left(\nu_j(x)\right)\right)\right) : x \in \Delta(s) \right\}. \quad (3.23)$$

□

4 Fundamental tools II

In this section, we construct the last five fundamental tools, which are related to the Puiseux series expansion of each irreducible branch of an analytic curve:

Proposition 4.1 *Let*

$$\psi = \sum_{n=0}^{\infty} c_n z^n \quad \in \mathbb{C}\{z\} \quad (4.1)$$

be a power series with radius of convergence $\rho \in]0, \infty]$, *and let* $D :=]-\rho, \rho[$. *Then, the following are equivalent:*

(i) *There exists a nonempty subset* A *of* D *such that* A *is not a discrete subset of* D *and such that* $\psi(A) \subset \mathbb{R}$.
(ii) $\psi(D) \subset \mathbb{R}$.
(iii) $c_n \in \mathbb{R}$ *for all* $n \in \mathbb{Z}_0^+$.

Proof Define $f : D \to \mathbb{R}$ by

$$f(x) = \mathrm{Im}\,(\psi(x)) = (1/(2i))\left(\psi(x) - \psi(x)^*\right). \quad (4.2)$$

Then, it is easily established that f has a power series expansion with $\sum_{n=0}^{\infty} \mathrm{Im}\,(c_n)\, z^n$ whose radius of convergence is obviously not less than ρ:

$$f(x) = \sum_{n=0}^{\infty} \mathrm{Im}\,(c_n)\, x^n \quad (4.3)$$

where $x \in D$. Hence, f is real analytic on D.

Bearing in mind the fact that D is a connected subset of $\mathbb{R}$, notice that the following statements are equivalent:

(i)′ There exists a nonempty subset A of D such that A is not a discrete subset of D and such that $f(A) = \{0\}$.
(ii)′ $f(D) = \{0\}$.
(iii)′ $\mathrm{Im}\,(c_n) = 0$ for all $n \in \mathbb{Z}_0^+$.

On the other hand, clearly we have: (i) $\Leftrightarrow$ (i)′, (ii) $\Leftrightarrow$ (ii)′, and (iii) $\Leftrightarrow$ (iii)′. Therefore, statements (i), (ii), and (iii) are equivalent. □

Proposition 4.2 *Let*

$$\psi = \sum_{n=0}^{\infty} c_n z^n \in \mathbb{C}\{z\} \tag{4.4}$$

be a power series with radius of convergence $\rho \in]0, \infty]$*, let* $w \in \mathbb{C}$ *with* $|w| = 1$*, and let* $r \in]0, \rho[$*. Let*

$$L(w, r) := \{w\xi : \xi \in [0, r]\}. \tag{4.5}$$

Suppose that

$$\psi(L(w, r)) \subset \mathbb{R}. \tag{4.6}$$

Then, we have

$$c_n w^n \in \mathbb{R} \tag{4.7}$$

for all $n \in \mathbb{Z}_0^+$.

Proof Let $w \in \mathbb{C}$ with $|w| = 1$ and consider the power series

$$\psi_1 = \sum_{n=0}^{\infty} c_n w^n z^n, \tag{4.8}$$

whose radius of convergence is obviously ρ. Next, note that the identity

$$\psi(L(w, r)) = \psi_1(L(1, r)) \tag{4.9}$$

holds and hence we have

$$\psi_1(L(1, r)) \subset \mathbb{R}. \tag{4.10}$$

Since $L(1, r)$ is a non-discrete subset of $]-\rho, \rho[$, we may apply Proposition 4.1 to the power series ψ_1 and get $c_n w^n \in \mathbb{R}$ for all $n \in \mathbb{Z}_0^+$. □

Proposition 4.3.A *Let* $k \in \mathbb{Z}^+$ *and let*

$$\wp = \sum_{n=0}^{\infty} c_n z^{n/k} (c_n \in \mathbb{C}) \tag{4.11}$$

be a Puiseux series with radius of convergence $s \in]0, \infty]$*. Let* $r \in]0, s[$*.*

Consider the following conditions:

(a1) $\wp(x)$ *is real for any* $x \in [0, r]$*.*
(a2) $\wp(x)$ *is real for any* $x \in [-r, r]$*.*

(b1) *For all* $n \in \mathbb{Z}_0^+$, $c_n \in \mathbb{R}$.
(b2) *For all* $n \in \mathbb{Z}_0^+$, *if* $2n/k \notin \mathbb{Z}_0^+$, *then* $c_n = 0$.
(b3) *For all* $n \in \mathbb{Z}_0^+$, *if* $n/k \notin \mathbb{Z}_0^+$, *then* $c_n = 0$.

Then, we have:

(i) (*a1*) $\Rightarrow$ (*b1*) *and (b2),*
(ii) (*a2*) $\Rightarrow$ (*b1*) *and (b3).*

Remark 4.1 The evaluation of the above Puiseux series $\wp$ at $x \in [-r, r]$ is considered as a multi-valued function (cf. Concluding Remarks given at the end of this section). Proposition 4.3.A is equivalent to the following Proposition 4.3.B, which is formulated within the context of a power series.

Proposition 4.3.B *Let* $k \in \mathbb{Z}^+$ *and let*

$$\psi = \sum_{n=0}^{\infty} c_n z^n \quad \in \mathbb{C}\{z\} \tag{4.12}$$

be a power series with radius of convergence $\rho \in]0, \infty]$ *and let* $r \in \left]0, \rho^k\right[$, *where* ρ^k *denotes* ∞ *if* $\rho = \infty$.

Consider the following conditions:

(a1) $\psi(x)$ *is real for any* $x \in \bigcup_{\theta \in [0,r]} \{t \in \mathbb{C} : t^k = \theta\}$.
(a2) $\psi(x)$ *is real for any* $x \in \bigcup_{\theta \in [-r,r]} \{t \in \mathbb{C} : t^k = \theta\}$.
(b1) *For all* $n \in \mathbb{Z}_0^+$, $c_n \in \mathbb{R}$.
(b2) *For all* $n \in \mathbb{Z}_0^+$, *if* $2n/k \notin \mathbb{Z}_0^+$, *then* $c_n = 0$.
(b3) *For all* $n \in \mathbb{Z}_0^+$, *if* $n/k \notin \mathbb{Z}_0^+$, *then* $c_n = 0$.

Then, we have:

(i) (*a1*) $\Rightarrow$ (*b1*) *and (b2),*
(ii) (*a2*) $\Rightarrow$ (*b1*) *and (b3).*

Proof Let L be as in Proposition 4.2.
Assume (a1). Then, we have

$$\psi\left(L\left(\exp(2\pi j i/k), r^{1/k}\right)\right) \subset \mathbb{R} \tag{4.13}$$

for all $j \in \{1, \ldots, k\}$. By Proposition 4.2, this implies that

$$c_n \exp(2\pi j n i/k) \in \mathbb{R} \tag{4.14}$$

for all $j \in \{1, \ldots, k\}$ and $n \in \mathbb{Z}_0^+$. Setting $j = k$, we have

$$c_n \exp(2\pi n i/k) = c_n \in \mathbb{R} \tag{4.15}$$

for all $n \in \mathbb{Z}_0^+$. Thus, (a1) $\Rightarrow$ (b1).

By (4.14) and (4.15), we have

$$\begin{aligned}\mathrm{Im}\,(c_n \exp\,(2\pi jni/k)) &= c_n \mathrm{Im}\,(\exp\,(2\pi jni/k))\\ &= c_n\, \sin\,(2\pi jn/k)\\ &= 0 \end{aligned} \tag{4.16}$$

for all $j \in \{1, \ldots, k\}$ and $n \in \mathbb{Z}_0^+$. Setting $j = 1$, we then have

$$c_n \sin\,(\pi\,(2n/k)) = 0 \tag{4.17}$$

for all $n \in \mathbb{Z}_0^+$. Since the set of zeros of $\sin\,(\pi\,(\cdot)) : \mathbb{R}_0^+ \to \mathbb{R}$ is $\mathbb{Z}_0^+$, if $2n/k \notin \mathbb{Z}_0^+$ then $c_n = 0$ for all $n \in \mathbb{Z}_0^+$. Thus, we have verified that (a1) $\Rightarrow$ (b2).

Assume (a2). Since (a2) $\Rightarrow$ (a1), we have only to show (b3).
Notice that (a2) implies that

$$\psi\left(L\left(\exp\,(\pi i/k), r^{1/k}\right)\right) \subset \mathbb{R} \tag{4.18}$$

By Proposition 4.2, we have

$$c_n \exp(\pi ni/k) \in \mathbb{R} \tag{4.19}$$

for all $n \in \mathbb{Z}_0^+$. Recall $c_n \in \mathbb{R}$ for all $n \in \mathbb{Z}_0^+$, and observe that

$$\begin{aligned}\mathrm{Im}\,(c_n \exp\,(\pi ni/k)) &= c_n \mathrm{Im}\,(\exp\,(\pi ni/k))\\ &= c_n\, \sin\,(\pi n/k)\\ &= 0 \end{aligned} \tag{4.20}$$

for all $n \in \mathbb{Z}_0^+$. Since the set of zeros of $\sin\,(\pi\,(\cdot)) : \mathbb{R}_0^+ \to \mathbb{R}$ is $\mathbb{Z}_0^+$, if $n/k \notin \mathbb{Z}_0^+$ then $c_n = 0$ for all $n \in \mathbb{Z}_0^+$. Thus, we have verified that (a2) $\Rightarrow$ (b3). □

Proposition 4.4 *Let $s \in \mathbb{R}^+$, let $k \in \mathbb{Z}^+$, and let $h \in H_0\left(\Delta\left(s^{1/k}\right)\right)$. Let*

$$A := \bigcup_{j=1}^{k} \Gamma(h \circ \nu_j), \tag{4.21}$$

where $\nu_j \colon \Delta(s) \to \Delta\left(s^{1/k}\right)$ is the function defined by $\nu_j(x) = \hat{\nu}_{k,j}(x)$. Then, the following statements are equivalent:

(i) *There exists an $\varepsilon \in \mathbb{R}^+$ such that*

$$A \cap (]-\varepsilon, \varepsilon[\, \times (\mathbb{C} - \mathbb{R})) = \emptyset. \tag{4.22}$$

(ii) *There exists an $h_0 \in H_r(-s, s)$ such that for all $j \in \{1, \ldots, k\}$ and $x \in\,]-s, s[$*

$$h_0(x) = \left(h \circ \nu_j\right)(x) \tag{4.23}$$

and such that

$$\Gamma\,(h_0) = A \cap (]-s, s[\, \times \mathbb{R}) = A \cap (]-s, s[\, \times \mathbb{C}) \tag{4.24}$$

(*iii*) *The equality*

$$A \cap (]-s, s[\, \times (\mathbb{C} - \mathbb{R})) = \emptyset \tag{4.25}$$

holds.

Proof (i) $\Rightarrow$ (ii): Assume (i) and let

$$\psi = \sum_{n=0}^{\infty} c_n z^n \quad \in \mathbb{C}\{z\} \tag{4.26}$$

be a power series with radius of convergence $\rho \geq s^{1/k}$ with

$$\psi(x) = h(x) \tag{4.27}$$

for all $x \in \Delta\left(s^{1/k}\right)$.

First, note that the following relations hold for all $j \in \{1, \ldots, k\}$:

$$\begin{aligned} \emptyset &= A \cap (]-\varepsilon, \varepsilon[\, \times (\mathbb{C} - \mathbb{R})) \\ &\supset \Gamma\left(h \circ \nu_j\right) \cap (]-\varepsilon, \varepsilon[\, \times (\mathbb{C} - \mathbb{R})) = \emptyset, \end{aligned} \tag{4.28}$$

so that we have

$$\bigcup_{j=1}^{k} \left(h \circ \nu_j\right) \cap (]-\varepsilon, \varepsilon[) \subset \mathbb{R}, \tag{4.29}$$

and hence

$$\bigcup_{j=1}^{k} h\left(\nu_j\,(]-\varepsilon, \varepsilon[)\right) = h\left(\bigcup_{j=1}^{k} \nu_j\,(]-\varepsilon, \varepsilon[)\right) \subset \mathbb{R}. \tag{4.30}$$

Second, recalling the definition of ν_j, notice that

$$\bigcup_{j=1}^{k} \nu_j\,(]-\varepsilon, \varepsilon[) = \bigcup_{\theta \in]-\varepsilon, \varepsilon[} \left\{t \in \mathbb{C} : t^k = \theta\right\}. \tag{4.31}$$

So, by (4.27), (4.30), and (4.31), we see that

$$\psi\left(\bigcup_{j=1}^{k} \nu_j\,(]-\varepsilon, \varepsilon[)\right) = \psi\left(\bigcup_{\theta \in]-\varepsilon, \varepsilon[} \left\{t \in \mathbb{C} : t^k = \theta\right\}\right) \subset \mathbb{R}, \tag{4.32}$$

and we can apply Proposition 4.3.B(ii) to get the following statements:

(a) For all $n \in \mathbb{Z}_0^+$, $c_n \in \mathbb{R}$.
(b) For all $n \in \mathbb{Z}_0^+$, if $n/k \notin \mathbb{Z}_0^+$, then $c_n = 0$.

Hence, we have for all $j \in \{1, \ldots, k\}$ and $x \in \Delta(s)$

$$\psi\left(\nu_j(x)\right) = \sum_{n=0}^{\infty} c_n \left(\nu_j(x)\right)^n = \sum_{n=0}^{\infty} c_{kn} \left(\nu_j(x)\right)^{kn} = \sum_{n=0}^{\infty} c_{kn} x^n \tag{4.33}$$

Note that the Cauchy-Hadamard theorem implies that the radius of convergence of the power series $\sum_{n=0}^{\infty} c_{kn} z^n$ is not less than s. Equalities (4.27) and (4.33) and statements (a) and (b) now imply that $h_0 :]-s, s[\to \mathbb{R}$ defined by

$$h_0(x) = \sum_{n=0}^{\infty} c_{kn} x^n \tag{4.34}$$

satisfies the following two conditions:

$$h_0 \in H_r(-s, s), \tag{4.35}$$
$$h_0(x) = \left(h \circ \nu_j\right)(x) \tag{4.36}$$

for all $j \in \{1, \ldots, k\}$ and $x \in]-s, s[$.
Moreover, by (4.35) and (4.36), we infer that

$$A \cap (]-s, s[\times \mathbb{C}) = \bigcup_{j=1}^{k} \left(\Gamma\left(h \circ \nu_j\right) \cap (]-s, s[\times \mathbb{C})\right) = \bigcup_{j=1}^{k} \left(\Gamma\left(h \circ \nu_j\right) \cap (]-s, s[\times \mathbb{R})\right) = A \cap (]-s, s[\times \mathbb{R}) = \bigcup_{j=1}^{k} \Gamma(h_0) = \Gamma(h_0). \tag{4.37}$$

Thus, (i) implies (ii).
(ii) $\Rightarrow$ (iii): Note that (4.24) implies that

$$\begin{aligned}\emptyset &= (A \cap (]-s,s[\times \mathbb{C})) - (A \cap (]-s,s[\times \mathbb{R}))\\ &= A \cap ((]-s,s[\times \mathbb{C}) - (]-s,s[\times \mathbb{R}))\\ &= A \cap (]-s,s[\times (\mathbb{C} - \mathbb{R}))\,. \end{aligned} \tag{4.38}$$

Hence, (ii) clearly implies (iii).

(iii) $\Rightarrow$ (i): This is trivially true. □

The fundamental tools I and II established in this article form the basis of the forthcoming part IV of this series, which together with its preceding parts will play an important role in the Second Generation Fukui Project.

Concluding remarks Let X and Y be nonempty sets. A nonempty set $R \subset X \times Y$ is said to be a relation. A set $F \subset X \times Y$ is said to be a "function" if for any $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in F$.

Let us call a nonempty set $M \subset X \times Y$ a multi-function-set of $X \times Y$, if there exist a finite number of "functions" $F_1, \ldots, F_k \subset X \times Y$ such that

$$M = \bigcup_{j=1}^{k} F_j.$$

Now let $k \in \mathbb{Z}^+$ and let

$$\wp = \sum_{n=0}^{\infty} c_n z^{n/k} \quad (c_n \in \mathbb{C})$$

be a Puiseux series with radius of convergence $s \in\,]0, \infty[$. Consider the function $h \in H_0\left(\Delta\left(s^{1/k}\right)\right)$ defined by $h(x) = \sum_{n=0}^{\infty} c_n x^n$. Let

$$M_\wp := \bigcup_{j=1}^{k} \Gamma\left(h \circ \nu_j\right),$$

where $\nu_j : \Delta(s) \to \Delta\left(s^{1/k}\right)$ is the function defined by $\nu_j(x) = \hat{\nu}_{k,j}(x)$.

Then, $M_\wp$ is a multi-function-set of $\Delta(s) \times \mathbb{C}$ associated with the above Puiseux series $\wp$. In the forthcoming part IV of this series, we shall exploit multi-function-sets to investigate the sets of zeros of locally defined analytic functions.

Acknowledgments Special thanks are due to Prof. M. Spivakovsky and Prof. K.F. Taylor for providing the author with valuable comments on the manuscript of this article. Thanks are also due to Prof. K. Saito, Institute for the Physics and Mathematics of the Universe, and Prof. I. Naruki for their valuable suggestions that were important to form this series of papers.

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